Intelligent Agents

Probability & Bayesian Reasoning

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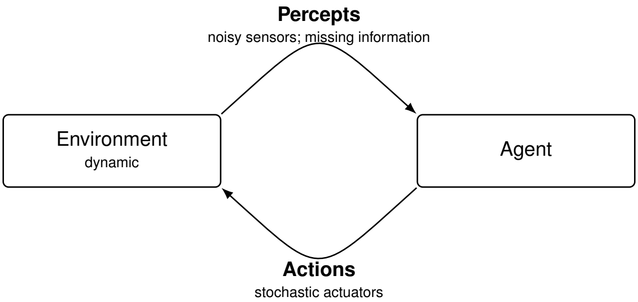
Motivation — Why Uncertainty?

- AI systems operate in noisy, unpredictable environments.
- Sensors fail, information is incomplete; actions have uncertain results; environments are dynamic.
- We need a principled framework to model and reason under uncertainty.



Limitations of Deterministic Logic

- Logic uses crisp truth values (True/False) — no notion of likelihood.
- Ambiguity and multiple causes break brittle rules.
- Example: "fever $\Rightarrow$ flu" ignores colds, COVID, etc.

Truth Table: (fever) $\Rightarrow$ (flu)

fever	flu	fever $\Rightarrow$ flu
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

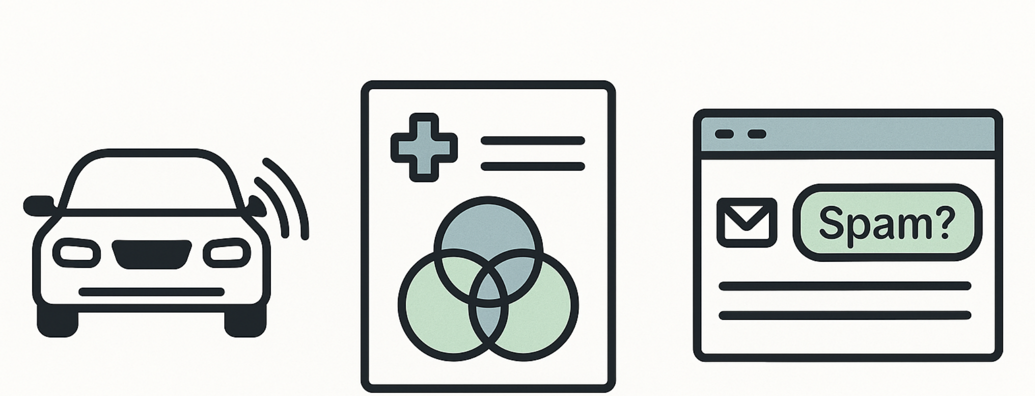
Example Conditional Probabilities

(Disease given fever is True/False)

	P(Flu $\mid \cdot$)	P(Cold $\mid \cdot$)	P(COVID $\mid \cdot$)
Fever = True	0.40	0.20	0.25
Fever = False	0.03	0.15	0.02

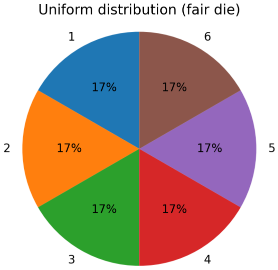
Real-World Uncertainty

- Self-driving cars: imperfect sensors, unpredictable agents.
- Medical diagnosis: overlapping symptoms, imperfect tests.
- Spam filtering: unusual but benign messages.



Foundations of Probability

- **Random variable:** A variable whose values come from a *sample space* of possible outcomes.
- **Sample space** (Ω): The complete set of all possible outcomes.
- **Probability distribution** P: Assigns a likelihood to each outcome in Ω.
- **Axioms of probability:**
  - Non-negativity: $P(A) \geq 0$
  - Normalization: $P(\Omega) = 1$
  - Additivity (disjoint): If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
- **Events:** Subsets of the sample space (e.g., “die shows an even number”).
- **Bounds:** $0 \leq P(A) \leq 1$.



Uniform distribution over outcomes 1–6

Two Random Variables: Rolling Two Dice

- **Setup:** Roll two fair six-sided dice. Define random variables X (die 1) and Y (die 2).
- **Sample space:** $\Omega = \{1,\ldots,6\} \times \{1,\ldots,6\}$
  All ordered pairs (i,j), total of 36 outcomes.
- **Joint distribution (uniform):**
  $P(X = i, Y = j) = \frac{1}{36}$ for all $i,j \in \{1,\ldots,6\}$.
- **Event example (sum = 7):**
  $A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
  $P(A) = \frac{6}{36} = \frac{1}{6}$.

Two Dice: Full Sample Space

Sample space: $\Omega = \{(i, j) : i, j \in \{1, \ldots, 6\}\}$

$$\begin{array}{c}
(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)
\end{array}$$

Total: $|\Omega| = 36$ equally likely outcomes ($P = 1/36$ each).

Two Dice: Grouping by Sum

Event grouping: outcomes with the same total.

$$
\begin{aligned}
\text{Sum } 2 &: (1,1) && 1 \text{ outcome}\\
\text{Sum } 3 &: (1,2),(2,1) && 2 \text{ outcomes}\\
\text{Sum } 4 &: (1,3),(2,2),(3,1) && 3 \text{ outcomes}\\
\text{Sum } 5 &: (1,4),(2,3),(3,2),(4,1) && 4 \text{ outcomes}\\
\text{Sum } 6 &: (1,5),(2,4),(3,3),(4,2),(5,1) && 5 \text{ outcomes}\\
\text{Sum } 7 &: (1,6),(2,5),(3,4),(4,3),(5,2),(6,1) && 6 \text{ outcomes}\\
\text{Sum } 8 &: (2,6),(3,5),(4,4),(5,3),(6,2) && 5 \text{ outcomes}\\
\text{Sum } 9 &: (3,6),(4,5),(5,4),(6,3) && 4 \text{ outcomes}\\
\text{Sum } 10 &: (4,6),(5,5),(6,4) && 3 \text{ outcomes}\\
\text{Sum } 11 &: (5,6),(6,5) && 2 \text{ outcomes}\\
\text{Sum } 12 &: (6,6) && 1 \text{ outcome}
\end{aligned}
$$

Joint, Marginal, and Conditional

- **Joint:** $P(A, B)$ — probability that A and B both occur.
- **Marginal:** $P(A) = \sum_b P(A, b)$ (discrete) or $\int P(A, b)\, db$.
- **Conditional:** $P(A|B) = \dfrac{P(A, B)}{P(B)}$ (if $P(B) > 0$).

	Heavy	Light	Total
Sunny	0.20	0.30	**0.50**
Rainy	**0.15**	0.20	**0.35**
Snowy	0.05	0.10	**0.15**
Total	**0.40**	**0.60**	**1.00**

Joint $P(\text{Rainy}, \text{Heavy})$
Marginals $P(\text{Weather})$, $P(\text{Traffic})$
Conditional $P(\text{Traffic} \mid \text{Sunny})$

Joint Probability: Two Dice

X = Die 1, Y = Die 2 Sample space Ω of 36 equally likely outcomes

$$p_{X,Y}(x,y) = \Pr(X = x,\ Y = y) = \tfrac{1}{36} \text{ for all } x, y \in \{1, \ldots, 6\}$$

	$y{=}1$	$y{=}2$	$y{=}3$	$y{=}4$	$y{=}5$	$y{=}6$
$x{=}1$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}2$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}3$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}4$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}5$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}6$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

The joint distribution assigns a probability to each ordered pair (x, y). For fair, independent dice, all 36 outcomes have probability $1/36$.

Marginal Probability from the Joint

$p_X(x) = \sum_y p_{X,Y}(x,y)$ and $p_Y(y) = \sum_x p_{X,Y}(x,y)$

	$y{=}1$	$y{=}2$	$y{=}3$	$y{=}4$	$y{=}5$	$y{=}6$	$p_X(x)$
$x{=}1$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{6}{36}{=}\frac{1}{6}$
$x{=}2$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$x{=}3$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$x{=}4$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$x{=}5$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$x{=}6$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{6}$
$p_Y(y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Marginals are row/column sums of the joint. With fair, independent dice, $p_X(x) = p_Y(y) = \frac{1}{6}$ uniformly.

Conditional Probability: Given the Sum is 7

$\Pr(X = x \mid S{=}7) = \dfrac{\Pr(X = x,\ S{=}7)}{\Pr(S{=}7)}$ where $S = X + Y$

	$y{=}1$	$y{=}2$	$y{=}3$	$y{=}4$	$y{=}5$	$y{=}6$
$x{=}1$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}2$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}3$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}4$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}5$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
$x{=}6$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

Event $S = 7$ (highlighted): $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ — 6 outcomes.

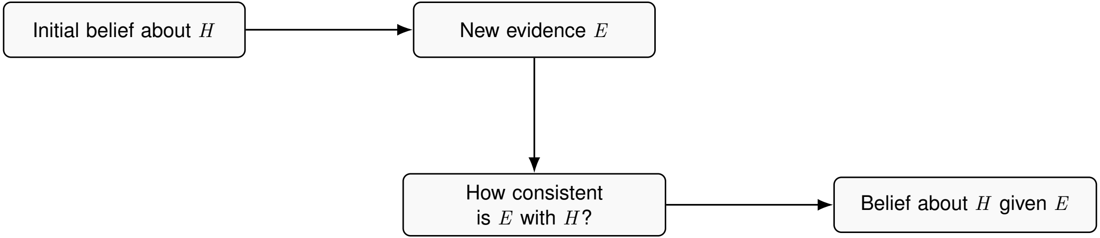
$$\Pr(S{=}7) = \frac{6}{36} = \frac{1}{6}.$$

For any $x \in \{1, \ldots, 6\}$, the compatible y is $7{-}x$, so:

$$\Pr(X{=}x \mid S{=}7) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}.$$

Thus $X \mid (S{=}7)$ is uniform on $\{1, \ldots, 6\}$.

Bayes' Rule — Intuition



Legend: H = Hypothesis E = Evidence

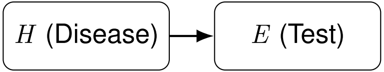
Bayes' Rule — Definition

$$P(H \mid E) = \frac{P(E \mid H)\, P(H)}{P(E)} \quad \text{with} \quad P(E) = \sum_h P(E \mid h) P(h)$$

- **Posterior** $P(H|E)$: belief after seeing evidence E.
- **Prior** $P(H)$: belief before seeing E.
- **Likelihood** $P(E|H)$: how compatible E is with H.
- **Evidence** $P(E)$: normalizer across all hypotheses.

Bayes in Action: Diagnostic Reasoning

Prior: $P(H)$ CPT: $P(E \mid H)$



Random variables

- $H \in \{\text{disease}, \neg\text{disease}\}$
- $E \in \{\text{positive}, \text{negative}\}$

Prior on H

$$P(H{=}\text{disease}) = p, \qquad P(H{=}\neg\text{disease}) = 1 - p$$

Likelihoods (test characteristics)

	E=positive	E=negative
H=disease	s	$1 - s$
H=¬disease	$1 - t$	t

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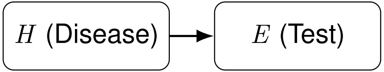

Given p, s, t, compute $P(H \mid E)$ by Bayes' rule.

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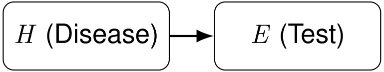

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$P(E = \text{positive} \mid H = \text{disease}) =$

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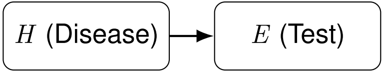

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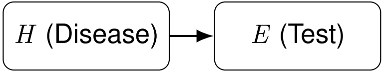

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$P(E = \text{positive} \mid H = \text{disease}) =$

$P(H = \text{disease}) =$

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Bayes in Action: Diagnostic Reasoning

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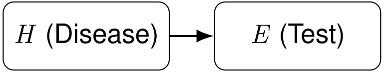

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$P(E = \text{positive} \mid H = \text{disease}) =$

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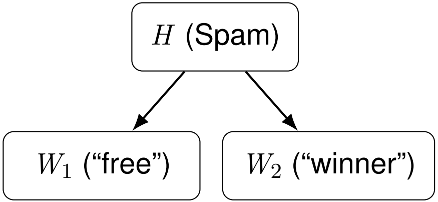
$P(E = \text{positive}) =$

$P(H = \text{disease} \mid E = \text{positive}) =$

Bayes in Action: Spam Filtering

Prior: $P(H)$



CPTs: $P(W_i \mid H)$

Prior on H

$$P(H{=}\text{spam}) = \gamma, P(H{=}\text{ham}) = 1 - \gamma$$

Random variables

- $H \in \{\text{spam, ham}\}$
- $W_1 \in \{\text{present, absent}\}$ (e.g., `free`)
- $W_2 \in \{\text{present, absent}\}$ (e.g., `winner`)

Naive Bayes CPTs (feature conditionals)

	present	absent
$W_1 \mid H{=}\text{spam}$	α_1	$1 - \alpha_1$
$W_1 \mid H{=}\text{ham}$	β_1	$1 - \beta_1$

	present	absent
$W_2 \mid H{=}\text{spam}$	α_2	$1 - \alpha_2$
$W_2 \mid H{=}\text{ham}$	β_2	$1 - \beta_2$

Bayes in Action: Spam Filtering

Trickier Computation

$$P(H{=}\text{spam} \mid w_1, w_2) =$$